# Tail Asymptotics under Beta Random Scaling

ENKELEJD HASHORVA [1] and ANTHONY G. PAKES [2]

May 13, 2009



**Abstract:** Let $X, Y, B$ be three independent random variables such that $X$ has the same distribution function as $YB$. Assume that $B$ is a beta random variable with positive parameters $\alpha, \beta$ and $Y$ has distribution function $H$ with $H(0) = 0$. Pakes and Navarro (2007) show under some mild conditions that the distribution function $H_{\alpha,\beta}$ of $X$ determines $H$. Based on that result we derive in this paper a recursive formula for calculation of $H$, if $H_{\alpha,\beta}$ is known. Furthermore, we investigate the relation between the tail asymptotic behaviour of $X$ and $Y$. We present three applications of our asymptotic results concerning the extremes of two random samples with underlying distribution functions $H$ and $H_{\alpha,\beta}$, respectively, and the conditional limiting distribution of bivariate elliptical distributions.

*Key words and phrases:* Beta random scaling; fractional integral; elliptical distribution; max-domain of attraction; asymptotics of sample maxima; conditional limiting results, estimation of conditional distribution; Weibull-tail distribution; Gardes-Girard estimator.

## 1 Introduction

Let $X, Y, B$ be three independent random variables such that

$$X \stackrel{d}{=} BY, \tag{1.1}$$

where $\stackrel{d}{=}$ stands for equality of the distribution functions. In our context the random variable $B$ plays the role of a random scaling or multiplier. Clearly, if the distribution functions of $Y$ and $B$ are known, then the distribution function of $X$ can be easily determined. In various theoretical and practical situations the question of interest is whether the distribution function of $Y$ can be determined provided that those of $X$ and $B$ are known. Indeed, random scaling of $Y$ by $B$ is treated in several papers and different contexts, see for instance the recent contributions Tang and Tsitsiashvili (2003,2004), Jessen and Mikosch(2006), Tang (2006,2008), Pakes (2007), Pakes and Navarro (2007), Beutner and Kamps (2008a,b).

Unless otherwise stated, in this article we fix $B$ to be a beta random variable with positive parameters $\alpha, \beta$. If $H$ denotes the distribution function of $Y$, then the distribution function of $X$ (denoted by $H_{\alpha,\beta}$) is defined in terms of $H$ and both parameters $\alpha, \beta$. If $Y$ is another beta random variable, then $X$ is the product of two such beta random variables, which have been studied extensively in the literature, see Galambos and Simonelli (2004), Nadarajah (2005), Nadarajah and Kotz (2005b, 2006), Dufresne (2007), Beutner and Kamps (2008a) and the references therein.

Our main impetus for dealing with the beta random scaling comes from Pakes and Navarro (2007) which paves the way for the distributional and asymptotic considerations in this paper. Theorem 2.2 therein gives an explicit formula for the calculation of the distribution function $H$, provided that $H_{\alpha,\beta}$ satisfies some weak growth restrictions on its derivatives. Utilising the aforementioned theorem, we show in this paper that the distribution function $H$ can be calculated iteratively without imposing any additional assumption on $H_{\alpha,\beta}$.

[1] Department of Mathematical Statistics and Actuarial Science, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland, hashorva@stat.unibe.ch

[2] School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, W. A., 6009, Australia, E-mail: pakes@maths.uwa.edu.au

This iterative inversion may lack the elegance of the explicit formula in Pakes and Navarro (2007), but it turns out to be quite useful in asymptotic contexts where we can define the tail behaviour of the survivor function of $Y$ when that of the survivor function of $X$ is known, and vice-versa.

We present three applications of our asymptotic results:
a) Determining which maximal domain of attraction contains $H_{\alpha,\beta}$ when the membership of $H$ is known;
b) The derivation of conditional limiting results for bivariate elliptical random vectors; and
c) New estimators for the conditional distribution function and the conditional quantile function of bivariate elliptical random vectors allowing one component of the random vector to grow to infinity.

The paper is organized as follows. In the next section we give some preliminary results. The main result of Section 3 is the iterative inversion for $H_{\alpha,\beta}$ – Theorem 3.3 below. In Section 4 we investigate the asymptotic relation of the survivor function of $X$ and $Y$ under conditions arising in extreme value theory, showing in particular that $H$ is attracted to an extreme value distribution if and only if $H_{\alpha,\beta}$ is attracted to the same distribution. The direct implications are formulated (in Section 7) in a generality which subsumes the particular case of beta scaling. Conditional limiting results and estimation of conditional distribution function for bivariate elliptical random vectors is discussed in Sections 5 and 6. All proofs and some related results are relegated to Section 7.

## 2 Preliminaries

We introduce notation and then discuss some properties of the Weyl fractional-order integral operator. A key result of Pakes and Navarro (2007) is recalled because it is crucial for our considerations.

We use notation such as $X \sim F$ to mean that $X$ is a random variable with distribution function $F$, and $\overline{F} := 1 - F$ denotes the corresponding survivor function. The upper endpoint of the distribution function $F$ is denoted by $r_F$ and its lower endpoint by $l_F$. If $\alpha, \beta > 0$ then beta$(\alpha,\beta)$ and gamma$(\alpha,\beta)$ denote respectively the beta and the gamma distributions with corresponding density functions

$$(B(\alpha,\beta))^{-1}x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1), \qquad \text{and } \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta x), \quad x \in (0,\infty),$$

where $B(\alpha,\beta)$ is the beta function and $\Gamma(\alpha)$ is the gamma function. Since beta distributed random variables appear below in several instances, we use exclusively the notation $B_{\alpha,\beta}$ for a beta random variable with parameters $\alpha, \beta$. On occasion it is convenient to extend the definition to understand $\boldsymbol{P}\{B_{0,\beta} = 0\} = 1$ if $\beta > 0$ and $\boldsymbol{P}\{B_{\alpha,1} = 1\} = 1$ if $\alpha > 0$. Unless otherwise stated, factors in products of random variables are assumed to be independent.

Next, define the Weyl fractional-order integral operator $I_\beta, \beta > 0$ by

$$(I_\beta h)(x) \quad := \quad \frac{1}{\Gamma(\beta)} \int_x^\infty (y-x)^{\beta-1} h(y)\, dy, \quad x > 0, \tag{2.1}$$

with $h : [0,\infty) \to I\!R$ a measurable function. The function $I_\beta h$ is well defined if (see Pakes and Navarro (2007))

$$\int_\varepsilon^\infty x^{\beta-1}|h(x)|\, dx < \infty$$

is satisfied for all $\varepsilon > 0$, in which case we write $h \in \mathcal{I}_\beta$ with the understanding that $\beta$ may assume negative values. Define further (consistently) $I_0 h := h$. If $h$ is a density function of a positive random variable $Y \sim H$, then $I_\beta h$ is well-defined for every $\beta > 0$. Suppose $g$ is a measurable function such that if $Y \sim H$, then $\boldsymbol{E}\{Y^{\beta-1}|g(Y)|\} < \infty$. Then we define

$$(\mathcal{J}_{\beta,g} H)(x) \quad = \quad \frac{1}{\Gamma(\beta)} \int_x^{r_H} (y-x)^{\beta-1} g(y)\, dH(y), \quad \forall x \in (l_H, r_H), \tag{2.2}$$

i.e., $\mathcal{J}_{\beta,g}$ denotes the Weyl-Stieltjes fractional-order integral operator acting on the class of distribution functions on $I\!R$ with weight function $g$.

The Weyl fractional-order integral operator is closely related to beta random scaling. To see this, let $\alpha, \beta > 0$ and $Y > 0$ and $B_{\alpha,\beta}$ be independent random variables such that

$$X \quad \stackrel{d}{:=} \quad Y B_{\alpha,\beta}, \qquad \text{where } X \sim H_{\alpha,\beta}, Y \sim H, \tag{2.3}$$

and $l_H \geq 0$. In the light of equation (14) in Pakes and Navarro (2007), for any $x \in (l_H, r_H)$ we have

$$H_{\alpha,\beta}(x) \quad = \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} (I_\beta p_{-\alpha-\beta} H)(x), \tag{2.4}$$

with $p_s$ the power function defined by

$$p_s(x) := x^s, \quad s \in I\!R, \ x > 0.$$

We mention in passing two important topics in probability theory and statistical applications where the Weyl fractional-order integral operator is encountered: a) the sized- or length biased law (see e.g., Pakes (2007), Pakes and Navarro (2007)); and b) the Wicksell problem (see e.g., Reiss and Thomas (2007)). For the essentials of fractional integrals and derivatives see Miller and Ross (1993).

Now we state three properties of$I_\beta h$.

**Lemma 2.1.** *Let $\beta, c$ be positive constants, and let $h$ be a real measurable function.*
*i) If $h \in \mathcal{I}_{\beta+c}$, then*

$$I_\beta I_c h = I_c I_\beta h = I_{\beta+c} h. \tag{2.5}$$

*ii) Let $D^n$ denote the $n$-fold derivative operator ($n \in I\!N$). If the $n$-fold derivative $h^{(n)} := D^n h$ exists almost everywhere and $h^{(n)} \in \mathcal{I}_\beta$, then*

$$D^n I_\beta h \quad = \quad I_\beta h^{(n)} \tag{2.6}$$

*and*

$$D^k I_n \quad = \quad (-1)^k I_{n-k}, \quad k = 1, \ldots, n. \tag{2.7}$$

*iii) If $\lambda \in (0, \beta)$ and $H$ is a distribution function on $I\!R$ with $H(0) = 0$, then*

$$(I_{\beta-\lambda} p_{-\beta} (I_\lambda p_{-\alpha-\lambda} H))(x) \quad = \quad x^{-\lambda} (I_\beta p_{-\alpha-\beta} H)(x), \quad \forall x \in (0, \infty). \tag{2.8}$$

The next theorem, which is an insignificant variation of Theorem 2.2 in Pakes and Navarro (2007) shows that the survivor function $\overline{H}$ can be retrieved by applying the differential and the Weyl fractional-order integral operator to $\overline{H}_{\alpha,\beta}$.

**Theorem 2.2.** *Let $H$,$H_{\alpha,\beta}, \alpha, \beta \in (0, \infty)$ be as above, with $H_{\alpha,\beta}(0) = 0$. If $H_{\alpha,\beta}^{(n-1)}$ is absolutely continuous and $H_{\alpha,\beta}^{(n-i)} \in \mathcal{I}_{\delta-\alpha-i}, i = 0, \ldots, n$ with $\delta$ and $n$ such that*

$$\beta + \delta =: n \in I\!N, \quad \delta \in [0, 1), \tag{2.9}$$

*then*

$$\overline{H}(x) \quad = \quad (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta} (I_\delta D^n p_{-\alpha} \overline{H}_{\alpha,\beta})(x) \tag{2.10}$$

*holds for any $x \in (0, r_H)$.*

## 3 Iterative Calculation of $H$

Let $X, Y, B_{\alpha,\beta}$, related by (2.3), be as above. In this section our main interest is the determination of $H$ from the known form of $H_{\alpha,\beta}$. As already mentioned, an explicit formula is presented as Theorem 2.2 in Pakes and Navarro (2007) (see (2.10) above). If $\beta \in (0,1]$, then the only requirement for the validity of their theorem is that $H_{\alpha,\beta}(0) = 0$, which obviously is fulfilled whenever $H(0) = 0$. The following well-known multiplicative property of beta random variables is the key to our iterative version of Theorem 2.2 above. Specifically, if $\lambda \in (0, \beta)$, then

$$B_{\alpha,\beta} \stackrel{d}{=} B_{\alpha,\lambda} B_{\alpha+\lambda,\beta-\lambda}.$$

Consequently, (2.3) implies that

$$X \stackrel{d}{=} Y B_{\alpha,\beta} \quad \stackrel{d}{=} \quad Y B_{\alpha,\lambda} B_{\alpha+\lambda,\beta-\lambda}. \tag{3.1}$$

Theorem 2.2 of Pakes and Navarro (2007) and (3.1)implies the following result:

**Theorem 3.1.** *Let $\alpha, \beta$ be two positive constants, and let $X, Y, B_{\alpha,\beta}$ be independent random variables satisfying* (2.3) *with $X \sim H_{\alpha,\beta}, Y \sim H$, and $H(0) = 0$.*

*i) If $\lambda \in (0, \beta)$, then*

$$\overline{H}_{\alpha,\beta}(x) \quad = \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha+\lambda} (I_{\beta-\lambda} p_{-\beta} (I_\lambda p_{-\alpha-\lambda} \overline{H}))(x), \quad \forall x \in (0, r_H). \tag{3.2}$$

*ii) If $\beta - \lambda \in [0,1)$, and $\delta \in [0,1)$ is such that $\beta - \lambda + \delta = 1$, then*

$$\overline{H}_{\alpha,\lambda}(x) \quad = \quad \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta} \Big[ (\alpha+\lambda)(I_\delta p_{-\alpha-\lambda-1} \overline{H}_{\alpha,\beta})(x) + (\mathcal{J}_{\delta, p_{-\alpha-\lambda}} H_{\alpha,\beta})(x) \Big], \quad \forall x \in (0, r_H). \tag{3.3}$$

We state next a simple corollary which is of some interest in the context of the Weyl fractional-order integral operator.

**Corollary 3.2.** *Let $H$ be a distribution function on $I\!R$ such that $H(0) = 0$. Then for any $x \in (0, r_H)$ we have*

$$x^{\alpha-1} (\mathcal{J}_{\beta, p_{-\alpha-\beta+1}} H)(x) \quad = \quad (D p_\alpha I_\beta p_{-\alpha-\beta} H)(x) = -(D p_\alpha I_\beta p_{-\alpha-\beta} \overline{H})(x). \tag{3.4}$$

*Moreover, if $H$ possesses the density function $h$, then*

$$(I_\beta p_{-\alpha-\beta+1} h)(x) \quad = \quad \alpha (I_\beta p_{-\alpha-\beta} H)(x) + x (I_\beta D(p_{-\alpha-\beta} H))(x), \quad x \in (0, r_H). \tag{3.5}$$

The main result of this section is the following iterative formula for computing $H$ when $H_{\alpha,\beta}$ is known.

**Theorem 3.3.** *Let $X \sim H_{\alpha,\beta}, Y \sim H$ and $B_{\alpha,\beta}, \alpha, \beta > 0$ be three independent random variables satisfying* (2.3) *such that $H_{\alpha,\beta}(0) = 0$. If $\beta_0 := \beta > \beta_1 > \cdots > \beta_k > \beta_{k+1} := 0$, with $k \in \{0, I\!N\}$ and $\delta_i, i \le k+1$ are constants such that*

$$\lambda_i := \beta_{i-1} - \beta_i \in (0,1], \quad \delta_i := 1 - \lambda_i, \quad i = 1, \ldots, k+1, \tag{3.6}$$

*then we can construct distribution functions $H_0 := H, H_1, \ldots, H_{k+1} = H_{\alpha,\beta}$ such that*

$$\overline{H}_{i-1}(x) \quad = \quad \frac{\Gamma(\alpha+\beta_i)}{\Gamma(\alpha+\beta_{i-1})} x^{\alpha+\beta_{i-1}} \Big[ (\alpha+\beta_i)(I_{\delta_i} p_{-\alpha-\beta_i-1} \overline{H}_i)(x) + (\mathcal{J}_{\delta_i, p_{-\alpha-\beta_i}} H_i)(x) \Big], \quad \forall x \in (0, r_H). \tag{3.7}$$

**Remark 3.4.** *(a) Let $B_i \sim B_{\alpha_i,\beta_i}, i \ge 1$ be independent beta random variables and independent of $Y \sim H$. If the random variable $X$ with distribution function $H_n, n \ge 2$ has the stochastic representation*

$$X \stackrel{d}{=} Y \prod_{i=1}^{n} B_i^{c_i}, \quad c_i \in (0, \infty), \quad i = 1, \ldots, n, \tag{3.8}$$

*then Theorem 3.3 implies that $H$ can be retrieved recursively from $H_n$, provided that $H_n(0) = 0$.*

*(b) An interesting (open) question arises in connection with random products. Specifically, if $\mathcal{N}$ is a counting random variable taking positive integer values independent of $Y, B_i, i \geq 1$, such that*

$$X \stackrel{d}{=} Y \prod_{i=1}^{\mathcal{N}} B_i^{c_i} \quad \text{where } X \sim H_{\mathcal{N}}, \tag{3.9}$$

*then under what conditions on $\mathcal{N}$ can we (recursively) compute the distribution function $H$ if $H_{\mathcal{N}}$ is known? Also arises a similar question if $X, Y$ are related by*

$$X \stackrel{d}{=} Y[B_3 B_1 + (1 - B_3) B_2]. \tag{3.10}$$

# 4 Tail Asymptotics

The tail asymptotics of products have been studied in papers such as Berman (1983, 1992), Cline and Samorodnitsky (1994), Tang and Tsitsiashvili (2003, 2004), Jessen and Mikosch (2006), Tang (2006, 2008), and the references therein. Our asymptotic considerations below can be motivated by considering sample maxima.

Specifically, let $X_i, Y_i, i = 1, \ldots, n$, be independent copies of $X = Y B_{\alpha,\beta}$ and $Y$, respectively, and

$$M_{X,k} := \max_{1 \leq j \leq k} X_j, \quad M_{Y,k} := \max_{1 \leq j \leq k} Y_j, \quad k \geq 1$$

be the corresponding sample maxima. From extreme value theory (see e.g., de Haan and Ferreira (2006), Falk et al. (2004, p. 23), Resnick (1987, p. 38)) if there are constants $a_n > 0, b_n$ such that

$$\lim_{n \to \infty} \sup_{t \in I\!R} \Big| H^n(a_n t + b_n) - Q(t) \Big| = 0, \tag{4.1}$$

then we have the convergence in distribution

$$(M_{Y,n} - b_n)/a_n \stackrel{d}{\to} \mathcal{M}_Y \sim Q, \quad n \to \infty, \tag{4.2}$$

where $Q$ is a univariate extreme value distribution (Gumbel, Fréchet or Weibull). If (4.1) holds (write $H \in MDA(Q)$) it is of some interest to investigate the asymptotic behaviour of $M_{X,k}, k \geq 1$, where $X_i, i \leq n$ are the results of a beta random scaling i.e.,

$$X_i \stackrel{d}{=} Y_i B_i, \quad B_i \stackrel{d}{=} B_{\alpha,\beta}, \quad i = 0, \ldots, n, \quad n \geq 1, \tag{4.3}$$

with $Y_i \sim H$ and the $B_i$'a and $Y_i$'s mutually independent. Thus $X_i \sim H_{\alpha,\beta}$. A key question is whether $H_{\alpha,\beta}$ is in a maximal domain of attraction if $H$ is, and conversely? We answer this below, as well as exposing the explicit tail asymptotic relations underlying (4.3).

## 4.1 Gumbel Max-domain of Attraction

If (4.1) holds with $Q = \Lambda$ the unit Gumbel distribution ($\Lambda(x) := \exp(-\exp(-x)), x \in I\!R$), then there exists a positive measurable *scaling* function $w$ (see e.g., de Haan and Ferreira (2006), Resnick (1987, p. 46)) such that

$$\lim_{x \uparrow r_H} \frac{\overline{H}(x + t/w(x))}{\overline{H}(x)} = \exp(-t), \quad \forall t \in I\!R \tag{4.4}$$

is valid. We write $H \in MDA(\Lambda, w)$ if (4.4) holds. The scaling function $w$ satisfies

$$\lim_{x \uparrow r_H} x w(x) = \infty, \quad \text{and } \lim_{x \uparrow r_H} w(x)(r_H - x) = \infty, \quad \text{if } r_H < \infty, \tag{4.5}$$

and also the self-neglecting property

$$\lim_{x\uparrow r_H} \frac{w(x+t/w(x))}{w(x)} = 1, \tag{4.6}$$

which holds locally uniformly for $t \in I\!\!R$; see e.g., Resnick (1987, p. 41). Note that most authors work with the so-called auxiliary function $1/w(x)$, but our convention follows Berman (1992) because results we prove are closely linked to some in his Chapter 12.

Canonical examples of distribution functions in the Gumbel max-domain of attraction are the univariate Gaussian and the gamma distributions, which are special cases of distribution functions whose scaling functions have the form (for $x$ large)

$$w(x) \quad = \quad \frac{r\theta x^{\theta-1}}{1+L_1(x)}, \tag{4.7}$$

where $L_1(x)$ is regularly varying at infinity with index $\theta\mu, \mu \in (-\infty, 0)$ and $r, \theta$ are positive constants. Note that $\theta = 2$ for the Gaussian case, and we have for the gamma$(\alpha, \beta)$ case that $\theta = 1$, $w(x) = \beta$ and

$$\lim_{x\to\infty} \frac{\overline{H}(x+t)}{\overline{H}(x)} \quad = \quad \exp(-\beta t), \quad \forall t \in I\!\!R. \tag{4.8}$$

Distribution functions $H$ that satisfy (4.8) comprise what in other contexts is called the exponential tail class $\mathcal{L}(\beta)$. See Pakes (2004) for references, and Pakes and Steutel (1997) where they are called medium-tailed.

We state now the first result of this section, a close relative of Theorem 12.3.1 in Berman (1992); see Example 1 below for the latter. In §7 we will state and prove the general proposition Theorem 7.4 which subsumes both direct assertions.

**Theorem 4.1.** *Let $H, H_{\alpha,\beta}$ be as in Theorem 3.3. Then $H \in MDA(\Lambda, w)$ iff (if and only if) $H_{\alpha,\beta} \in MDA(\Lambda, w)$. If one of these holds, then*

$$\overline{H}_{\alpha,\beta}(x) \quad = \quad (1+o(1))K(xw(x))^{-\beta}\overline{H}(x), \quad x \uparrow r_H, \tag{4.9}$$

*where $K := \Gamma(\alpha+\beta)/\Gamma(\alpha)$, and the density function $h_{\alpha,\beta}$ of $H_{\alpha,\beta}$ satisfies*

$$\lim_{x\uparrow r_H} \frac{h_{\alpha,\beta}(x)}{w(x)\overline{H}_{\alpha,\beta}(x)} \quad = \quad 1. \tag{4.10}$$

The asymptotic equivalence (4.9) is the principal assertion here, as can be seen by noting that if one of the distribution functions $F$ and $H$ is in $MDA(\Lambda, w)$ and they are related by

$$\overline{F}(x) = (1+o(1))x^c(w(x))^\mu \overline{H}(x), \qquad (x \uparrow r_H), \tag{4.11}$$

where $c, \mu$ are real, then it follows from (4.4) and (4.6) that the other distribution function is in $MDA(\Lambda, w)$.

It is well-known that if $H$ is a univariate distribution function with upper endpoint $r_H = \infty$ and $H \in MDA(\Lambda, w)$, then $\overline{H}$ is rapidly varying (see Resnick (1987)) i.e.,

$$\lim_{x\to\infty} \frac{\overline{H}(cx)}{\overline{H}(x)} \quad = \quad 0, \quad \forall c > 1. \tag{4.12}$$

A necessary ingredient in the proof of Theorem 4.1 is the following rate of convergence refinement to (4.12); recall the first member of (4.5).

**Lemma 4.2.** *Let $H$ be a univariate distribution function with $r_H = \infty$. If $H \in MDA(\Lambda, w)$, then we have for any constant $\mu \geq 0$*

$$\lim_{x\to\infty} (xw(x))^\mu \frac{\overline{H}(cx)}{\overline{H}(x)} \quad = \quad 0, \quad \forall c > 1. \tag{4.13}$$

**Remark 4.3.** *(a) The self-neglecting property* (4.6) *implies that the density function* $h_{\alpha,\beta}$ *of* $H_{\alpha,\beta}$ *satisfies*

$$\frac{h_{\alpha,\beta}(x+t/w(x))}{h_{\alpha,\beta}(x)} \quad \to \quad \exp(-t), \quad x \uparrow r_H$$

*locally uniformly for* $t \in \mathbb{R}$*, provided that either* $H \in MDA(\Lambda, w)$*, or* $H_{\alpha,\beta} \in MDA(\Lambda, w)$*.*

*(b) By Theorem 4.1, if* $H_{\alpha,\beta} \in MDA(\Lambda, w)$*, then we can reverse* (4.9) *obtaining*

$$\overline{H}(x) \quad = \quad (1+o(1))\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(xw(x))^{\beta}\overline{H}_{\alpha,\beta}(x), \quad x \uparrow r_H. \tag{4.14}$$

*See Berman (1992) and Hashorva (2007d) for similar results. Further note that* (4.13) *and* (4.14) *imply for any* $c \in (1,\infty)$ *that*

$$\overline{H}(x) = o(\overline{H}_{\alpha,\beta}(cx)), \quad \overline{H}_{\alpha,\beta}(x) = o(\overline{H}(cx)), \text{ and } (xw(x))^{\beta}\overline{H}_{\alpha,\beta}(x) = o(1), \quad x \uparrow r_H.$$

We give next two illustrations of Theorem 7.4.

**Example 1.** (a) Theorem 12.3.1 in Berman (1992) follows from Theorem 7.4(a) by taking (see (7.11))

$$\phi(u) = \boldsymbol{P}\{\sqrt{1-B_{\alpha,\beta}} > u\}$$

and checking that, since $1 - B_{\alpha,\beta} \stackrel{d}{=} B_{\beta,\alpha}$, (7.11) holds with $C = 2^{\alpha}/\alpha B(\alpha,\beta)$ *and* the exponent $\beta$ replaced with $\alpha$.

(b) Let $H, F$ be two distribution functions as in Theorem 4.1 and suppose that $l_H = 0$ and $r_H = \infty$. We assume that the random multiplier $B$ has the stochastic representation

$$B \stackrel{d}{=} \lambda U_1 + (1-\lambda)U_2, \quad \lambda \in (0,1),$$

where $U_1, U_2$ are two independent positive random variables such that for $i = 1, 2$

$$\boldsymbol{P}\{U_i > 1-s\} = (1+o(1))c_i s^{d_i}, \quad c_i, d_i \in (0,\infty), \quad s \downarrow 0.$$

It follows that as $s \downarrow 0$

$$\boldsymbol{P}\{B > 1-s\} \quad = \quad (1+o(1))\frac{c_1 c_2}{\lambda^{d_1}(1-\lambda)^{d_2}}\frac{\Gamma(d_1+1)\Gamma(d_2+1)}{\Gamma(d_1+d_2+1)}s^{d_1+d_2}.$$

Further, assume for all large $x$ that

$$\overline{H}(x) \quad = \quad (1+o(1))Mx^N\exp(-rx^{\theta}), \quad M>0, r>0, \theta>0, N \in \mathbb{R}. \tag{4.15}$$

Since, for any $t \in \mathbb{R}$,

$$\lim_{x\to\infty}\frac{\overline{H}(x+tx^{1-\theta}/(r\theta))}{\overline{H}(x)} \quad = \quad \exp(-t)$$

we have $H \in MDA(\Lambda, w)$ with

$$w(x) \quad = \quad r\theta x^{\theta-1}, \quad x > 0. \tag{4.16}$$

In view of Theorem 7.4 the distribution function $F$ of $BY$ satisfies $F \in MDA(\Lambda, w)$ and, as $x \to \infty$,

$$\overline{F}(x) \quad = \quad (1+o(1))C^* x^{N-\theta(d_1+d_2)}\exp(-rx^{\theta}),$$

with

$$C^* \quad = \quad M(r\theta)^{-d_1-d_2}\frac{c_1c_2}{\lambda^{d_1}(1-\lambda)^{d_2}}\Gamma(d_1+1)\Gamma(d_2+1).$$

## 4.2 Regularly Varying Tails

We deal next with distribution functions $H_{\alpha,\beta}$ in either the Fréchet or the Weibull max-domains of attraction. As we will discuss below, the asymptotics of $H_{\alpha,\beta}$ when $H$ is attracted to the Fréchet distribution is quite well known, and results for the Weibull max-domain of attraction are less complete. In Section 7 we offer simpler proofs of these results, and their converses, i.e., when $H_{\alpha,\beta}$ belongs to one of these max-domains of attractions, then so does $H$.

The unit Fréchet distribution function with positive index $\gamma$ is $\Phi_\gamma(x) := \exp(-x^{-\gamma}), x > 0$. It is well-known that a distribution function $H$ with infinite upper endpoint $r_H = \infty$ is in the Fréchet max-domain of attraction (see e.g., Falk et al. (2004), Resnick (1987)) iff $\overline{H}$ is regularly varying at infinity with index $-\gamma < 0$, i.e.,

$$\lim_{x\to\infty} \frac{\overline{H}(xt)}{\overline{H}(x)} = t^{-\gamma}, \quad \forall t \in (0,\infty). \tag{4.17}$$

If $l_H = 0$ and $0 < \gamma < 1$, then this condition is the criterion that $H$ is attracted to a positive stable law with index $\gamma$. Breiman (1965, Proposition 3) shows that if this holds, then the distribution function $F$ of $X = BY$, where the random multiplier $B$ is independent of $Y$, is also attracted to the same positive stable law provided that $\boldsymbol{E}\{|B|\} < \infty$. (Thus $B$ is not restricted in sign or magnitude.) Specifically, $H$ and $F$ are tail equivalent, i.e.,

$$\overline{F}(x) = (1 + o(1))\boldsymbol{E}\{B^\gamma\}\overline{H}(x) \quad (x \to \infty). \tag{4.18}$$

Jessen and Mikosch (2006, p. 184) observe that Breiman's proof is valid for any positive $\gamma$ if $B \geq 0$ and $\boldsymbol{E}\{B^{\gamma+\epsilon}\} < \infty$ for some $\epsilon > 0$. Berman (1992, Theorem 12.3.2) proves this tail equivalence for the case $B = \sqrt{1 - B_{\alpha,\beta}}$.

So in particular, we conclude that if $\alpha, \beta > 0$ and $H_{\alpha,\beta}$, is defined via (2.3) with $H_{\alpha,\beta}(0) = 0$, then

$$\overline{H}_{\alpha,\beta}(x) \quad = \quad (1 + o(1))\boldsymbol{E}\{B^\gamma_{\alpha,\beta}\}\overline{H}(x), \quad x \to \infty, \tag{4.19}$$

and

$$\boldsymbol{E}\{B^\gamma_{\alpha,\beta}\} = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\alpha+\beta+\gamma)}.$$

The next theorem asserts that this tail equivalence holds also if $\gamma = 0$, and conversely, if $\gamma > 0$ and $H_{\alpha,\beta} \in MDA(\Phi_\gamma)$, then so is $H$. Breiman's methodology is completely analytical, and in Section 7 we shall give a much simpler proof for the case of a general bounded multiplier $0 \leq B \leq 1$. We indicate too how this can be extended to the general result.

**Theorem 4.4.** *Let $H, H_{\alpha,\beta}, \alpha, \beta > 0$ be two distribution functions defined via* (2.3) *with $H(0) = 0$. Then $H$ satisfies* (4.17) *with some $\gamma \geq 0$, iff $H_{\alpha,\beta}$ satisfies* (4.17) *with the same index $\gamma$. Furthermore, for any $\gamma > 0$ we have*

$$\lim_{x\to\infty} \frac{x h_{\alpha,\beta}(x)}{\overline{H}_{\alpha,\beta}(x)} \quad = \quad \gamma. \tag{4.20}$$

**Example 2.** Theorem 4.4 shows in particular that Pareto tails are preserved under independent beta random scaling.

The unit Weibull distribution function with index $\gamma > 0$ is $\Psi_\gamma(x) := \exp(-|x|^\gamma), x < 0$. It is well known that if $H$ has a finite upper endpoint (say $r_H = 1$), then $H \in MDA(\Psi_\gamma)$ iff

$$\lim_{x\downarrow 0} \frac{\overline{H}(1-tx)}{\overline{H}(1-x)} \quad = \quad t^\gamma, \quad \forall t > 0. \tag{4.21}$$

Theorem 12.3.3 in Berman (1992) is closely related to the following result, and in Section 7 we prove a general theorem which subsumes both direct assertions.

**Theorem 4.5.** *Let $H, H_{\alpha,\beta}, \alpha, \beta$ be as in Theorem 4.4. If $r_H = 1, H(0) = 0$ and (4.21) holds for some $\gamma \geq 0$, then $H_{\alpha,\beta} \in MDA(\Psi_{\beta+\gamma})$ and*

$$\overline{H}_{\alpha,\beta}(1-x) = (1+o(1))Kx^{\beta}\overline{H}(1-x), \quad x \downarrow 0, \tag{4.22}$$

*with $K := \Gamma(\alpha+\beta)\Gamma(\gamma+1)/(\Gamma(\alpha)\Gamma(\gamma+\beta+1))$.*
*Furthermore we have*

$$\lim_{x \downarrow 0} \frac{x h_{\alpha,\beta}(1-x)}{\overline{H}_{\alpha,\beta}(1-x)} = \beta + \gamma > 0. \tag{4.23}$$

*Conversely, if $H_{\alpha,\beta} \in MDA(\Psi_{\beta+\gamma}), \gamma \geq 0$, then (4.21) is satisfied.*

**Remark 4.6.** *(a) If (2.3) holds with $B_{\alpha,\beta} \sim \mathrm{gamma}(\alpha, \beta)$, then in Lemma 17 of Hashorva et al. (2007) it is shown that $\overline{H}$ satisfies (4.17) with some $\gamma \geq 0$, iff $\overline{H}_{\alpha,\beta}$ satisfies (4.17) with the same index $\gamma$ (see also Jessen and Mikosch (2006)).*

*(b) Under the Gumbel or the Weibull max-domain of attraction assumption on $H$ or $H_{\alpha,\beta}$ by (4.5) we have*

$$\lim_{x \uparrow r_H} \frac{\overline{H}_{\alpha,\beta}(x)}{\overline{H}(x)} = 0,$$

*whereas when $H$ or $H_{\alpha,\beta}$ are in the Fréchet max-domain of attraction the above limit is a positive constant.*

# 5 Conditional Limiting Results

Let the bivariate random vector $(O_1, O_2)$ be uniformly distributed on the unit circle, $R \sim H$ be independent of $(O_1, O_2)$, and let $(S_1, S_2) \stackrel{d}{=} R(O_1, O_2)$ be the corresponding bivariate (planar) spherical random vector. Finally, define the bivariate elliptical random vector

$$(U, V) \stackrel{d}{=} (S_1, \rho S_1 + \sqrt{1-\rho^2} S_2), \quad \rho \in (-1, 1). \tag{5.1}$$

Distributional properties of spherical and elliptical random vectors are studied by many authors, e.g., Cambanis et al. (1981), Fang et al. (1990), Kotz et al. (2000) and their references. Referring to Cambanis et al. (1981) we have

$$O_1^2 \stackrel{d}{=} O_2^2 \sim \mathrm{beta}(1/2, 1/2). \tag{5.2}$$

Basic asymptotic properties of spherical and elliptical random vectors can be derived utilising (5.1) and (5.2). One line of enquiry is to determine the asymptotic behaviour of the conditional distribution of $V - \rho U$ given an event constraining the values of $U$. For example, in several statistical applications (see Abdous et al. (2005)) the approximation of the conditional random variable

$$Z_x^* \stackrel{d}{=} (V - \rho x) | U > x, \quad x \in I\!R$$

is of some interest. Since $V - \rho U = \sqrt{1-\rho^2} S_2$, the outcome follows directly from Theorem 12.3.3 in Berman (1992), i.e., if $H \in MDA(\Lambda, w)$, then

$$c(x) Z_x^* \stackrel{d}{\to} \sqrt{1-\rho^2} Z \quad x \uparrow r_H, \tag{5.3}$$

where $c(x) := \sqrt{w(x)/x}, x > 0$, and $Z$ is a standard Gaussian random variable. Abdous et al. (2005) is an independent account. Theorem 5.1 below embellishes this outcome.

The point-wise conditioned random variable

$$Z_x \stackrel{d}{=} (V - \rho x)|U = x, \quad x \in I\!R$$

is a particular case of the conditional multivariate models introduced by Heffernan and Tawn (2004) for treating certain inference problems. They raise the general problem of conditional limit laws when one component of a random vector tends to infinity, and they give results for some particular parametric families. It is known that (Hashorva (2006), Corollary 3.1) that $Z_x$ has the same Gaussian limit law as $Z_x^*$, i.e.,

$$c(x)Z_x \stackrel{d}{\to} \sqrt{1-\rho^2}Z, \quad x \uparrow r_H. \tag{5.4}$$

We will prove that if $H$ is absolutely continuous then (5.4) holds in the stronger sense that the density functions converge. We prove in addition that both limit assertions hold assuming that the (marginal) distribution of $|U|$ is attracted to the Gumbel distribution. Finally, Hashorva and Kotz (2009) gives an account of these results based on the strong Kotz approximation.

**Theorem 5.1.** *Let $H, (U, V), \rho \in (-1, 1), c(x), Z_x, Z_x^*, x > 0$ be as above with $|U| \sim G$ and $G(0) = 0$. If $G \in MDA(\Lambda, w)$ or $H \in MDA(\Lambda, w)$, then (a), (5.3) is satisfied; and (b), (5.4) is satisfied if, in addition, $H$ is absolutely continuous.*

The proof of this theorem rests on a closure lemma for distributions attracted to the Gumbel distribution.

**Lemma 5.2.** *Let $0 \le X \sim F$, $p > 0$ be a constant, and denote the distribution function of $X^p$ by $F_p$. Then $F \in MDA(\Lambda, w)$ iff $F_p \in MDA(\Lambda, w_p)$ where*

$$w_p(x) = p^{-1}x^{(1/p)-1}w\left(x^{1/p}\right).$$

## 6 Estimation of Conditional Survivor and Quantile Function

For $i = 1, 2, \ldots$, let $(U_i, V_i)$ be independent copies of $(U, V)$ as defined in the previous section, and suppose too that $R \sim H \in MDA(\Lambda, w)$ with $r_H = \infty$. We are interested in the conditional survivor function

$$\Psi_x(y) \quad := \quad \boldsymbol{P}\{V > y|U > x\}, \quad x, y \in I\!R.$$

Estimation of the distribution function $1 - \Psi_x(y)$ when $x$ is large is discussed in detail by Abdous et al. (2007). As noted there, if $x$ is large there may be insufficient data available for the effective estimation of $\Psi_x(y)$. Similar difficulties apply for estimation of the inverse function (or conditional quantile function), $\Theta(x, \cdot), s \in (0, 1), x \in I\!R$ of $1 - \Psi_x(\cdot)$. The Gaussian approximation implied by Theorem 5.1 entails

$$\sup_{y \in I\!R}\Big|\Psi_x(y\sqrt{x/w(x)} + \rho x) - \Phi(y/\sqrt{1-\rho^2})\Big| \to 0, \quad x \to \infty, \tag{6.1}$$

where $\Phi$ is the standard Gaussian distribution function.

On this basis, Abdous et al. (2007) propose two estimators of $\Psi_x$. Theorem 5.1 implies that the Gaussian approximation in (6.1) is valid if we assume instead that $U \sim G \in MDA(\Lambda, w)$. For estimation purposes this fact is crucial because we can estimate $w$ based only on the random sample $U_1, \ldots, U_n$, or $V_1, \ldots, V_n$.

A non-parametric estimator $\hat{\rho}_n$ of $\rho$ is given by (see e.g., Li and Peng (2009)))

$$\hat{\rho}_n \quad := \quad \sin(\pi \hat{\tau}_n/2), \quad n > 1, \tag{6.2}$$

where $\hat{\tau}_n$ is the empirical estimator of Kendall's tau. Now, if $\hat{w}_n(x)$ is an estimator of the scaling function $w(x)$ (for all large $x$), then by the above approximation we can estimate $\Psi_x(y)$ by

$$\hat{\Psi}_{n,x}(y) \quad := \quad \overline{\Phi}\Big(\hat{h}_n(y - \hat{\rho}_n x)/(1-\hat{\rho}_n^2)^{1/2}\Big), \quad n > 1, \tag{6.3}$$

where $\hat{h}_n(x) := (\hat{w}_n(x)/x)^{1/2}, x > 0$. An estimator for the quantile function $\Theta$ is then given by

$$\hat{\Theta}_n(x,s) \quad = \quad \rho_n x + \sqrt{1-\hat{\rho}_n^2}\,\Phi^{-1}(s)/\hat{h}_n(x), \quad x > 0, s \in (0,1), \tag{6.4}$$

with $\Phi^{-1}$ the inverse of $\Phi$. Both of these estimators are consequences of the Gaussian approximation. However, our concern here is with estimation of $w$. Specifically, we assume that the scaling function $w$ satisfies (4.7) with positive constants $r, \theta$ and $L_1$ regularly varying with index $\theta\mu, \mu \in (-\infty, 0)$. It follows that (see Abdous et al. (2007))

$$\overline{G}(x) \quad = \quad \exp(-rx^{\theta}(1+L_2(x)) \tag{6.5}$$

holds for all large $x$, where $L_2$ is another regularly varying function with index $-\theta\mu$. This places $G$ in the class of Weibull-tail distributions, and $\theta^{-1}$ is the so-called Weibull tail-coefficient (see Gardes and Girard (2006), or Diebolt et al. (2007)). Canonical examples of Weibull-tail distributions are the Gaussian, gamma, and extended Weibull distributions. Next, define for $i = 1, \ldots, n$,

$$R_i^{(1)} := U_i, \quad R_{in}^{(2)} := \sqrt{U_i^2 + (V_i - \hat{\rho}_n U_i)^2/(1-\hat{\rho}_n^2)}$$

and write $R_{1:n}^{(k)} \le \cdots \le R_{n:n}^{(k)}, k = 1, 2$ for the associated order statistics. Based on $R_i^{(1)}, i \le n$ or $R_{in}^{(2)}, i \le n$ we may construct the Gardes-Girard (2006) estimator of $\theta$,

$$\hat{\theta}_n^{(j)} := \frac{1}{T_n}\frac{1}{k_n}\sum_{i=1}^{n}\Big(\log R_{n-i+1:n}^{(j)} - \log R_{n-k_n+1:n}^{(j)}\Big), \quad j = 1, 2,$$

where $1 \le k_n \le n, T_n > 0, n \ge 1$ are constants satisfying

$$\lim_{n\to\infty} k_n = \infty, \quad \lim_{n\to\infty}\frac{k_n}{n} = 0, \quad \lim_{n\to\infty}\log(T_n/k_n) = 1, \quad \lim_{n\to\infty}\sqrt{k_n}\,b(\log(n/k_n)) \to \lambda \in I\!R,$$

and the function $b$ (related to $L_1$) is regularly varying with index $\eta$. The scaling coefficient $r$ can be estimated by (see Abdous et al. (2007))

$$\hat{r}_n^{(j)} \quad = \quad \frac{1}{k_n}\sum_{i=1}^{k_n}\frac{\log(n/i)}{(R_{n-i+1:n}^{(j)})^{\hat{\theta}_n^{(j)}}}, \quad j = 1, 2, n > 1, \tag{6.6}$$

leading to the following estimators of $w$,

$$\hat{w}_n^{(j)}(x) \quad = \quad \hat{r}_n^{(j)}\hat{\theta}_n^{(j)}x^{\hat{\theta}_n^{(j)}-1}, \quad x > 0, j = 1, 2, n > 1. \tag{6.7}$$

Our suggestion is to estimate $w$ by $\hat{w}_n^{(1)}$, because it is based on independent and identically distributed $R_i, i \le n$. This differs from the estimator $\hat{w}_n^{(2)}$ recommended by Abdous et al. (2007) which is based on the dependent random variables $R_{1n}^{(2)}, \ldots, R_{nn}^{(2)}$ (recall $\hat{\rho}_n$ is estimated from $(U_i, V_i), i \ge 1$).

A third estimator of $w$ can be easily constructed by considering the sample $V_1, \ldots, V_n$ since by the assumption $U \stackrel{d}{=} V$.

Note in passing that if $\theta = 1$, then we have the estimator of $r$ (of interest for $G$ in $\mathcal{L}(r), r > 0$)

$$\hat{r}_n^{(1)} \quad = \quad \frac{1}{k_n}\sum_{i=1}^{k_n}\frac{\log(n/i)}{R_{n-i+1:n}^{(1)}}, \quad n > 1. \tag{6.8}$$

# 7 Further Results and Proofs

We present first some asymptotic results for the Weyl fractional-order integral operator, followed by the proofs of all the results in the previous sections.

**Theorem 7.1.** *Let $H$ be a univariate distribution function with $H(0) = 0$, $r_H \in (0, \infty]$, and $H \in MDA(\Lambda, w)$. If $\alpha$ is real and $\beta > 0$, then*

$$(\mathcal{J}_{\beta,p_{-\alpha}}H)(x) \quad = \quad (1 + o(1))(w(x))^{-(\beta-1)}x^{-\alpha}\overline{H}(x), \quad x \uparrow r_H, \tag{7.1}$$

*and*

$$(I_\beta p_{-\alpha}\overline{H})(x) \quad = \quad \frac{(1 + o(1))}{w(x)}(\mathcal{J}_{\beta,p_{-\alpha}}H)(x), \quad x \uparrow r_H. \tag{7.2}$$

Proof of Theorem 7.1 Let $W_x$ be a random variable whose survivor function is

$$\boldsymbol{P}\{W_x > z\} = \frac{\overline{H}(x + z/w(x))}{\overline{H}(x)}, \qquad (0 \le z < r(x)),$$

where

$$r(x) = (r_H - x)w(x) \text{ if } r_H < \infty, \quad \& \quad = \infty \text{ if } r_H = \infty.$$

Then (4.4) is equivalent to the convergence assertion $W_x \xrightarrow{d} W$ which has the standard exponential distribution. Observe now that for $x \in (0, r_H)$ we may write

$$\begin{aligned}(\mathcal{J}_{\beta,p_{-\alpha}}H)(x) \quad &= \quad \frac{1}{\Gamma(\beta)}\int_x^{r_H}(y - x)^{\beta-1}x^{-\alpha}\,dH(y) \\ &= \quad \frac{x^{-\alpha}\overline{H}(x)}{\Gamma(\beta)}\int_0^{r(x)}\left(\frac{z}{w(x)}\right)^{\beta-1}(1 + z/v(x))^{-\alpha}d_z\boldsymbol{P}\{W_x \le z\},\end{aligned}$$

where $v(x) = xw(x)$ and we have used the substitution $y = x + z/w(x)$ for the second equality. Hence

$$\frac{(w(x))^{\beta-1}x^\alpha}{\overline{H}(x)}(\mathcal{J}_{\beta,p_{-\alpha}}H) = \frac{1}{\Gamma(\beta)}\boldsymbol{E}\left\{W_x^{\beta-1}\left(1 + W_x/v(x)\right)^{-\alpha}\right\}.$$

It follows from (4.5) and the moment convergence theorem (Feller (1971, p. 252)) that the expectation converges to $\boldsymbol{E}\{W^{\beta-1}\} = \Gamma(\beta)$. This proves (7.1).

The same manoeuvres yield

$$\frac{x^\alpha(w(x))^\beta}{\overline{H}(x)}(I_\beta p_{-\alpha}\overline{H})(x) = \frac{1}{\Gamma(\beta)}\int_0^{r(x)} z^{\beta-1}(1 + z/v(x)^{-\alpha}\boldsymbol{P}\{W_x > z\}dz \to 1,$$

using the dominated convergence theorem, and (7.2) follows. □

Theorem 7.1 subsumes and generalizes results in Berman (1992, §12.2) applying to the case $r_H = \infty$. To align with Berman's notation, we use $\beta - 1$ to denote his parameter $p$, and in what follows we assume that $\boldsymbol{E}\{Y^\beta\} < \infty$.

(i) Propositions 12.2.3 and 4 in Berman (1992) concern distribution functions $F$ having the form

$$\overline{F}(x) = (1 + o(1))c\int_x^\infty (y - x)^{\beta-1}\overline{H}(y)dy.$$

It is easily seen that

$$\overline{F}(x) = (1 + o(1))c\Gamma(\beta)(\mathcal{J}_{\beta+1,p_0}H)(x) = (1 + o(1))c\Gamma(\beta)(w(x))^{-\beta}\overline{H}(x),$$

and this is valid if $\beta > 0$, which extends the range of parameter in Berman's Proposition 12.2.4.

(ii) Proposition 12.2.5 in Berman (1992) concerns survivor functions proportional to the order-$q$ stationary excess distribution generated by $H$, i.e.,

$$\overline{F}(x) = (1 + o(1))c\int_x^\infty y^{q-1}\overline{H}(y)dy,$$

where $q$ is real. The integral can be recast as

$$q^{-1}\int_x^\infty (y^q - x^q)\, dH(y) = q^{-1}x^q\overline{H}(x)\boldsymbol{E}\{(1+W_x/v(x))^q - 1\},$$

from which it follows that, as $x \to \infty$,

$$\overline{F}(x) = (1+o(1))cx^{q-1}\frac{\overline{H}(x)}{w(x)}.$$

(iii) The order-$q$ size-biased distribution generated by $H$ induces survivor functions of the form

$$\overline{F}(x) = (1+o(1))c\int_x^\infty y^q dH(y) = (1+o(1))c(\mathcal{J}_{1,p_q}H)(x) = (1+o(1))cx^q\overline{H}(x).$$

It follows from (4.11) that each above $F \in MDA(\Lambda, w)$.

Note that if $q > 0$, then the results under (ii) and (iii) are related *via* Theorem 4.1 because if $\widehat{Y}_q$ and $\widetilde{Y}_q$ denote the order-$q$ size-biased and stationary excess versions of $Y$, then $\widetilde{Y}_q \overset{d}{=} \widehat{Y}_q B_{q,1}$. See Pakes (1996, §4) for this connection and further generalization involving beta scaling.

**Theorem 7.2.** *Let $H$ be a univariate distribution function with $r_H = \infty$. Assume that $H(0) = 0$ and (4.17) holds with $\gamma \geq 0$. If $\beta > 0$ and $c$ are two constants such that $\beta + c < \gamma + 1$, then*

$$(\mathcal{J}_{\beta,p_c}H)(x) \quad = \quad (1+o(1))\frac{\gamma\Gamma(\gamma+1-\beta-c)}{\Gamma(\gamma+1-c)}\overline{H}(x)x^{\beta+c-1}, \quad x \to \infty. \tag{7.3}$$

*Furthermore if $\gamma \geq 0$, then*

$$(I_\beta p_c\overline{H})(x) \quad = \quad \frac{\Gamma(\gamma-\beta-c)}{\Gamma(\gamma-c)}\overline{H}(x)x^{\beta+c}, \quad x \to \infty. \tag{7.4}$$

Proof of Theorem 7.2 Let $W_x$ have the distribution function $\max(1-\overline{H}(xt)/\overline{H}(x), 0)$. Then (4.17) is equivalent to: If $\gamma > 0$, then $W_x \xrightarrow{d} W$ which has the Pareto survivor function $t^{-\gamma}$ for $t \geq 1$; and if $\gamma = 0$, then $W_x \xrightarrow{p} \infty$.

Substituting $y = tx$ into the integral defining $\mathcal{J}_{\beta,p_c}H$ gives the representation

$$(\mathcal{J}_{\beta,p_c}H)(x) = \frac{x^{\beta+c-1}\overline{H}(x)}{\Gamma(\beta)}\boldsymbol{E}\left[(W_x-1)^{\beta-1}W_x^c\right].$$

If $\gamma > 0$ and $\epsilon > 0$ is chosen so $\beta + c + \epsilon < \gamma + 1$, then $\boldsymbol{E}(W^{\beta+c+\epsilon-1}) < \infty$, and hence the above expectation converges to

$$\boldsymbol{E}\left[(W-1)^{\beta-1}W^c\right] = \gamma B(\gamma+1-\beta-c, \beta),$$

and (7.3) follows. This assertion follows too if $\gamma = 0$ because $(W_x-1)^{\beta-1}W_x^c < W_x^{\beta+c-1}$, and the exponent is negative.

The same substitution yields

$$(I_{\beta,p_c}\overline{H})(x) = \frac{\overline{H}(x)x^{\beta+c}}{\Gamma(\beta)}\int_1^\infty (t-1)^{\beta-1}t^c\boldsymbol{P}\{W_x > t\}dt,$$

and it is clear that the integral converges to

$$\int_1^\infty (t-1)^{\beta-1}t^{c-\gamma}dt = B(\gamma-\beta-c, \beta),$$

whence (7.4). □

**Theorem 7.3.** *Let $H$ be a univariate distribution function with upper endpoint $r_H = 1$. Assume that $H(0) = 0$, and that (4.21) holds with $\gamma \geq 0$. If $\beta > 0$ and $c \in I\!R$ are constants and $\gamma > 0$, then*

$$(\mathcal{J}_{\beta,p_c} H)(1-x) \quad = \quad (1+o(1))\frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma)}\overline{H}(1-x)x^{\beta-1}, \quad x \downarrow 0 \tag{7.5}$$

*and if $\gamma \geq 0$, then*

$$(I_\beta p_c \overline{H})(1-x) \quad = \quad (1+o(1))\frac{1}{\gamma+\beta}x(\mathcal{J}_{\beta,p_c} H)(1-x), \quad x \downarrow 0. \tag{7.6}$$

Proof of Theorem 7.3 Let $W_x \leq 1$ be a random variable having the distribution function $\overline{H}(1-tx)/\overline{H}(1-x)$. If $\gamma > 0$, then (4.21) is equivalent to $W_x \xrightarrow{d} W := U^{1/\gamma}$, where $U$ has the standard uniform distribution (i.e. beta$(1,1)$), and if $\gamma = 0$, then $W_x \xrightarrow{d} 1$. The substitution $y = 1 - xt$ yields

$$(\mathcal{J}_{\beta,p_c} H)(1-x) = \frac{\overline{H}(1-x)x^{\beta-1}}{\Gamma(\beta)}\boldsymbol{E}\{(1-W_x)^{\beta-1}(1-xW_x)^c\}.$$

If $\gamma > 0$, then the expectation converges as $x \downarrow$ to

$$\boldsymbol{E}\left\{(1-W)^{\beta-1}\right\} = \gamma B(\gamma, \beta),$$

and if $\gamma = 0$ then it converges to unity. So (7.5) follows in both cases.

The same substitution yields

$$(I_\beta p_c \overline{H})(1-x) = \frac{\overline{H}(1-x)x^\beta}{\Gamma(\beta)}\int_0^1 (1-t)^{\beta-1}(1-xt)^c \boldsymbol{P}\{W_x \leq t\}dt,$$

and the integral converges to $B(\gamma+1, \beta)$. Thus

$$(I_\beta p_c \overline{H})(1-x) = (1+o(1))\overline{H}(1-x)x^\beta \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)},$$

and (7.6) follows. □

Proof of Lemma 2.1 Since the first two statements are borrowed from Lemma 2.2 in Pakes and Navarro (2007) we show next only statement *iii*). Let $Y \sim H, B_{\alpha,\beta}, B_{\alpha,\lambda}$ and $B_{\alpha+\lambda,\beta-\lambda}$ be independent random variables. For any $\lambda \in (0, \beta)$ we have the stochastic representation (see (3.1))

$$YB_{\alpha,\beta} \quad \stackrel{d}{=} \quad Y^* B_{\alpha+\lambda,\beta-\lambda}, \quad Y^* \stackrel{d}{=} YB_{\alpha,\lambda},$$

with $Y^* \sim H_{\alpha,\lambda}$ another random variable independent of $B_{\alpha+\lambda,\beta-\lambda}$. Applying (2.4) we obtain for any $x \in (0, r_H)$

$$\begin{aligned} H_{\alpha,\beta}(x) \quad &= \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}x^\alpha (I_\beta p_{-\alpha-\beta} H)(x) \\ &= \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\lambda)}x^{\alpha+\lambda}(I_{\beta-\lambda} p_{-\alpha-\beta} H_{\alpha,\lambda})(x) \end{aligned}$$

and

$$H_{\alpha,\lambda}(x) \quad = \quad \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha)}x^\alpha (I_\lambda p_{-\alpha-\lambda} H)(x).$$

Consequently

$$(I_\beta p_{-\alpha-\beta} H)(x) \quad = \quad x^\lambda (I_{\beta-\lambda} p_{-\beta}(I_\lambda p_{-\alpha-\lambda} H))(x),$$

and the result follows. □

Proof of Theorem 2.2 The proof follows immediately from Theorem 2.2 in Pakes and Navarro (2007) and the identity

$$(I_{\beta,p_{-c}})(x) = \frac{\Gamma(c-\beta)}{\Gamma(c)} x^{\beta-c} \tag{7.7}$$

□

Proof of Theorem 3.1 The identity (7.7) implies that

$$1 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} (I_\beta p_{-\alpha-\beta})(x), \quad \forall x \in (0, r_H),$$

and hence (2.4)

$$\begin{aligned}
\overline{H}_{\alpha,\beta}(x) &= 1 - H_{\alpha,\beta}(x) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} (I_\beta p_{-\alpha-\beta})(x) - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} (I_\beta p_{-\alpha-\beta} H)(x) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} \Big[ (I_\beta p_{-\alpha-\beta})(x) - (I_\beta p_{-\alpha-\beta} H)(x) \Big] \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} (I_\beta p_{-\alpha-\beta} \overline{H})(x), \qquad (x \in (0, r_H))
\end{aligned}$$

thus the first result follows utilising further (2.8) which holds if $\overline{H}$ replaces $H$.

We show next the second claim. Since $H(0) = 0$, Lemma 2.1 in Pakes and Navarro (2007) shows that

$$H_{\alpha,\lambda}(0) = H_{\alpha,\beta}(0) = 0.$$

Furthermore, both $H_{\alpha,\lambda}$ and $H_{\alpha,\beta}$ are absolutely continuous and

$$X \stackrel{d}{=} Y^* B_{\alpha+\lambda,\beta-\lambda}, \qquad \text{with } Y^* \sim H_{\alpha,\lambda}, \quad X \sim H_{\alpha,\beta}.$$

Therefore, in order to show the proof we need to check the assumptions of Theorem 2.2. In our case $n = 1$, hence the condition $H^{(n-1)}_{\alpha,\beta} = H^{(0)}_{\alpha,\beta} = H_{\alpha,\beta}$ is absolutely continuous is satisfied. Since $H^{(1)}_{\alpha,\beta}$ is a density function and $\delta \in [0, 1)$, then clearly $H^{(1)}_{\alpha,\beta} \in \mathcal{I}_{\delta-\alpha-\lambda}$. Further we have $H^{(0)}_{\alpha,\beta} = H_{\alpha,\beta} \in \mathcal{I}_{\delta-\alpha-\lambda-1}$ since $H_{\alpha,\beta}$ is bounded by 1. Applying Theorem 2.2 for any $x \in (0, r_H)$ we may write

$$\begin{aligned}
\overline{H}_{\alpha,\lambda}(x) &= -\frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta} (I_\delta D(p_{-\alpha-\lambda} \overline{H}_{\alpha,\beta}))(x) \\
&= \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta} \Big[ (\alpha+\lambda)(I_\delta p_{-\alpha-\lambda-1} \overline{H}_{\alpha,\beta})(x) + (\mathcal{J}_{\delta,p_{-\alpha-\lambda}} H_{\alpha,\beta})(x) \Big],
\end{aligned}$$

and the result follows. □

Proof of Corollary 3.2 Letting $\lambda \to 0$ in (3.2) we obtain (recall $I_0 h := h$)

$$\overline{H}_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} (I_\beta p_{-\alpha-\beta} \overline{H})(x), \quad \forall x \in (0, r_H). \tag{7.8}$$

Consequently, we have

$$-h_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} (D(p_\alpha I_\beta p_{-\alpha-\beta} \overline{H}))(x), \quad \forall x \in (0, r_H)$$

and in view of (2.4),

$$h_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} (D(p_\alpha I_\beta p_{-\alpha-\beta} H))(x), \quad \forall x \in (0, r_H).$$

Since $h_{\alpha,\beta}$ is given by (see (22) in Hashorva et al. (2007))

$$h_{\alpha,\beta}(x) \quad = \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha-1} (\mathcal{J}_{\beta,p_{-\alpha-\beta+1}} H)(x), \quad \forall x \in (0, r_H), \tag{7.9}$$

the result follows. □

PROOF OF THEOREM 3.3 Let $B_{\alpha+\beta_i,\lambda_i} \sim \text{beta}(\alpha+\beta_i, \lambda_i), i = 0, \ldots, k$ be independent beta random variables independent of $X$ and $Y$. By the assumptions we may write

$$\begin{aligned} X &\stackrel{d}{=} Y B_{\alpha,\beta_0} \\ &\stackrel{d}{=} B_{\alpha,\beta_1} Y B_{\alpha+\beta_1,\beta_0-\beta_1} \\ &\stackrel{d}{=} Y_1 B_{\alpha,\beta_1}, \quad \text{with } Y_1 \stackrel{d}{=} Y_0 B_{\alpha+\beta_1,\beta_0-\beta_1} \stackrel{d}{=} Y_0 B_{\alpha+\beta_1,\lambda_1}, \quad Y_0 := Y. \end{aligned}$$

Similarly

$$X \quad \stackrel{d}{=} \quad Y_2 B_{\alpha,\beta_2}, \quad \text{with } Y_2 \stackrel{d}{=} Y_1 B_{\alpha+\beta_2,\lambda_2}$$

and repeating we arrive at

$$X \quad \stackrel{d}{=} \quad Y_k B_{\alpha,\beta_k}, \quad \text{with } Y_k \stackrel{d}{=} Y_{k-1} B_{\alpha+\beta_k,\lambda_k}.$$

Setting $Y_{k+1} := X$ we may write the above stochastic representation as

$$Y_{k+1} \quad \stackrel{d}{=} \quad Y_k B_{\alpha+\beta_{k+1},\lambda_{k+1}}.$$

Let $H_0 := H$ and $H_{k+1} := H_{\alpha,\beta}$. Applying (3.3) we obtain for any $i = 1, \ldots, k+1$,

$$\overline{H}_{i-1}(x) \quad = \tfrac{\gamma(\alpha+\beta_i)}{\gamma(\alpha+\beta_i+\lambda_i)} x^{\alpha+\beta_{i-1}} \Big[ (\alpha+\beta_i)(I_{\delta_i} p_{-\alpha-\beta_i-1} \overline{H}_i)(x) + (\mathcal{J}_{\delta_i,p_{-\alpha-\beta_i}} H_i)(x) \Big], \tag{7.10}$$

and the assertion follows. □

We precede our account of scaling relations for the Gumbel distribution with the following proof.
PROOF OF LEMMA 4.2 If $\beta > 0$ it follows from Theorem 7.1 that

$$\overline{H}_{1,\beta}(x) = (\mathcal{J}_{\beta+1,p_\beta})(x) = (1+o(1))\Gamma(1+\beta) \frac{\overline{H}(x)}{(xw(x))^\beta}, \qquad (x \uparrow r_H).$$

On the other hand, if $B := B_{1,\beta}$ and $c > 1$, then

$$\overline{H}_{1,\beta}(x) > \int_{cx}^\infty \boldsymbol{P}\{B > x/y\} dH(y) > \boldsymbol{P}\{B > c^{-1}\} \overline{H}(cx).$$

Combining these estimates yields

$$\limsup_{x\to\infty} (xw(x))^\beta \frac{\overline{H}(cx)}{\overline{H}(x)} < \infty.$$

The assertion (4.13) follows by choosing $\beta > \mu$ and appealing to (4.5) in the case $r_H = \infty$. □

The next result is the foreshadowed generalization of the direct assertion of Theorem 4.1. It comprises two parts which respectively yields a tail estimate of the distribution function of a random scaling, and its density function.

**Theorem 7.4.** *Suppose $H \in MDA(\Lambda, w)$. (a) If $\phi(u) \ge 0$ is defined and bounded on $[0,1]$ and it satisfies*

$$\phi(u) = (1+o(1))C(1-u)^\beta, \quad u \uparrow 1, \tag{7.11}$$

*where $\beta, C \ge 0$ are constants, then*

$$I(x) := \int_x^\infty \phi(x/y) dH(y) = (1+o(1))C\Gamma(1+\beta) \frac{\overline{H}(x)}{(xw(x))^\beta}, \quad x \uparrow r_H.$$

*(b) If $g(u) \geq 0$ is defined in $[0,1]$ such that $ug(u)$ is defined and bounded on $[0,u']$ for any $u' < 1$, and*

$$g(u) = (1+o(1))c(1-u)^{\beta-1} \quad u \uparrow 1, \tag{7.12}$$

*where $c \geq 0$ and $\beta > 0$ are constants, then*

$$J(x) := \int_x^{\infty} y^{-1} g(x/y) dH(y) = (1+o(1))c\Gamma(\beta) \frac{\overline{H}(x)}{x^{\beta}(w(x))^{\beta-1}}, \quad x \uparrow r_H.$$

Proof of Theorem 7.4 We prove only (b) since the details for (a) are similar and simpler. If $u' \in (0,1)$, then $x/y \leq u'$ if $y \geq x/u'$ and

$$J_1(x) := \int_{x/u'}^{\infty} y^{-1} g(x/y) dH(y) = O[x^{-1}\overline{H}(x/u')].$$

If $r_H$ is finite, then $J_1(x) = 0$ if $x > u' r_H$. If $r_H = \infty$, then, recalling that $v(x) = xw(x)$, Lemma 4.2 ensures that $J_1(x) = o\left(x^{-1}\overline{H}(x)(v(x))^{-\mu}\right)$ $(x \to \infty)$ for all positive $\mu$.

If $c$ is positive and $0 < \epsilon \ll c$, then it follows from (7.12) that $g(u)/(1-u)^{\beta-1} \in (c-\epsilon, c+\epsilon)$ if $u' < u < 1$ and $u'$ is sufficiently close to unity. Hence $J(x) - J_1(x)$ is asymptotically equal to

$$J_2(x) := c\int_x^{x/u'} y^{-1}(1-x/y)^{\beta-1} dH(y).$$

Proceeding as in the proof of Theorem 7.1 we obtain the representation

$$J_2(x) = \frac{c\overline{H}(x)}{x(v(x))^{\beta-1}} \boldsymbol{E}\left\{\frac{W_x^{\beta-1}}{(1+W_x/v(x))^{\beta}}; W_x \leq v(x)(1-u')/u'\right\}.$$

The expectation converges to $\boldsymbol{E}\{W^{\beta-1}\} = \Gamma(\beta)$. Taking $\mu > \beta$ above, we see that $J_1(x) = o(J_2(x))$, and the assertion follows. □

Proof of Theorem 4.1 Assume that $H \in MDA(\Lambda, w)$. The direct assertion (4.9) follow from Theorem 7.4(a) by setting $\phi(u) := \boldsymbol{P}\{B_{\alpha,\beta} > u\}$ and checking that (7.11) holds with $C = [\beta B(\alpha,\beta)]^{-1}$. Next, taking $g(u)$ as the density function of $B_{\alpha,\beta}$ it is obvious that the conditions of Theorem 7.4(b) are satisfied with $c = 1/B(\alpha,\beta)$. Thus (4.10) follows from (4.9) and (7.12).

To prove the converse, assume that $H_{\alpha,\beta} \in MDA(\Lambda, w)$ for some positive scaling function $w$. With the notation of Theorem 3.3 we may write for $i = 1, \ldots, k+1$

$$\overline{H}_{i-1}(x) = \frac{\Gamma(\alpha+\beta_i)}{\Gamma(\alpha+\beta_{i-1})} x^{\alpha+\beta_{i-1}} \Big[(\alpha+\beta_i)(I_{\delta_i} p_{-\alpha-\beta_i-1}\overline{H}_i)(x) + (\mathcal{J}_{\delta_i, p_{-\alpha-\beta_i}} H_i)(x)\Big], \quad \forall x \in (0, r_H), \tag{7.13}$$

where $\overline{H}_0 := \overline{H}, \overline{H}_{k+1} := \overline{H}_{\alpha,\beta}$. In view of Theorem 7.1 and (4.5), we obtain for $i = k+1$ that

$$\begin{aligned}\overline{H}_{i-1}(x) &= (1+o(1))\frac{\Gamma(\alpha+\beta_i)}{\Gamma(\alpha+\beta_{i-1})} x^{\alpha+\beta_{i-1}} (\mathcal{J}_{\delta_i, p_{-\alpha-\beta_i}} H_i)(x) \\ &= (1+o(1))\frac{\Gamma(\alpha+\beta_i)}{\Gamma(\alpha+\beta_{i-1})} x^{\beta_{i-1}-\beta_i} (w(x))^{-(\delta_i-1)}\overline{H}_i(x), \quad \forall x \uparrow r_H.\end{aligned}$$

By (4.5) and (4.6) it follows that $H_k \in MDA(\Lambda, w)$. Since the above holds for all $i = 1, \ldots, k$, it follows that $H_0 = H \in MDA(\Lambda, w)$ too. Next, (7.8) and (7.9) imply for any $x > 0$ that

$$\frac{h_{\alpha,\beta}(x)}{\overline{H}_{\alpha,\beta}(x)} = \frac{x(\mathcal{J}_{\beta, p_{-\alpha-\beta}} H)(x)}{(I_{\beta} p_{-\alpha-\beta-1} H)(x)}, \tag{7.14}$$

so applying Theorem 7.1 establishes (4.10), and the result follows. □

As foreshadowed above, the following argument includes a simple proof of (4.18) for an arbitrary bounded random scaling. We then show how this proof can be extended to remove the boundedness restriction.
PROOF OF THEOREM 4.4 With $W_x$ as in the proof of Theorem 7.2, clearly

$$\boldsymbol{P}\{XB > x\} = \overline{H}(x)\boldsymbol{E}\{\boldsymbol{P}\{B > W_x^{-1}\}\}.$$

But

$$\boldsymbol{P}\{B > W_x^{-1}\} \to \boldsymbol{P}\{B > W^{-1}\} = \int_0^1 \boldsymbol{P}\{W > u^{-1}\}d\boldsymbol{P}\{B \le u\} = \boldsymbol{E}\{B^\gamma\},$$

and hence $\overline{F}(x) = (1 + o(1))\overline{H}(x)\boldsymbol{E}\{B^\gamma\}$.

Similarly, if $B$ has the density function $g(u)$, then the density function of $F$ is

$$f(x) = x^{-1}\overline{H}(x)\boldsymbol{E}\{W_x^{-1}g(W_x^{-1})\}.$$

If $g$ satisfies appropriate boundedness conditions, which certainly are satisfied by beta density functions, then the expectation converges to

$$\boldsymbol{E}\left\{W^{-1}g(W^{-1})\right\} = \gamma\int_1^\infty w^{-\gamma-2}g(w^{-1})dw = \gamma\boldsymbol{E}\{B^\gamma\}.$$

It follows that $(h(x)/\overline{H}(x)) = (1 + o(1))(\gamma/x)$. The direct assertions of Theorem 4.4 follow.

The converse asserts that if $\overline{H}_{\alpha,\beta}$ is regularly varying with index $-\gamma \le 0$, then $\overline{H}$ is also regularly varying with index $-\gamma$. If $\gamma > 0$, then the proof follows from Theorem 3.3 and Theorem 7.2.

Alternatively, write can write $B_{\alpha,\beta} = Z_1/(Z_1 + Z_2)$, where $Y, Z_1, Z_2$, are independent random variables such that $Z_1 \sim$ gamma$(\alpha, 1)$ and $Z_2 \sim$ gamma$(\beta, 1)$. Since $Z_1 + Z_2 \sim$ gamma$(\alpha + \beta)$ is independent of $B_{\alpha,\beta}$, the relation $X \stackrel{d}{=} YB_{\alpha,\beta}$ is equivalent to $X(Z_1 + Z_2) \stackrel{d}{=} YZ_1$. It follows from Jessen and Mikosch (2006, Lemma 4.2(a)) that the survivor function of $YZ_1$ is regularly varying with index $-\gamma$, and Lemma 17 in Hashorva et al. (2007) implies the same is true for $\overline{H}(x)$. We emphasize that this proof is valid for $\gamma \ge 0$. □

Note that Theorem 12.3.2 in Berman (1992) follows from the above direct proof since

$$\boldsymbol{E}\{(1 - B_{\alpha,\beta})^{\gamma/2}\} = \boldsymbol{E}\{B_{\beta,\alpha}^{\gamma/2}\} = \frac{B(\alpha, \beta + \gamma/2)}{B(\alpha, \beta)}.$$

The situation where $B$ is allowed to be unbounded can be handled by writing

$$\boldsymbol{P}\{YB > x\} = \boldsymbol{P}\{YB > x; Y > x\} + \boldsymbol{P}\{YB > x; Y \le x\}. \tag{7.15}$$

Exactly as in the last proof, the first term on the right is asymptotically proportional to $\overline{H}(x)\boldsymbol{P}\{B > W^{-1}\}$, but now the probability term evaluates as

$$\boldsymbol{P}\{W > B^{-1}\} = \boldsymbol{P}\{B \ge 1\} + \boldsymbol{E}\{B^\gamma; B \le 1\}.$$

The second term on the right-hand side of (7.15) equals

$$\begin{aligned}\boldsymbol{P}\{x/B \le Y \le x, B > 1\} &= \int_1^\infty \left(\overline{H}(x/z) - \overline{H}(x)\right) d\boldsymbol{P}\{B \le z\} \\ &= (1 + o(1))\overline{H}(x)\left[\boldsymbol{E}\{B^\gamma; B > 1\} - \boldsymbol{P}\{B > 1\}\right],\end{aligned}$$

provided the limit here can be taken inside the integral. This is permissible if $\boldsymbol{E}\{B^{\gamma+\epsilon}\} < \infty$ for some $\epsilon > 0$.

The converse tail equivalence statement is open in general, but see Hashorva et al. (2007) for the case where $B$ has a gamma distribution.

The following result is the analogue of Theorem 7.4 for $H \in MDA(\Psi_\gamma)$ and it generalizes the direct assertion of Theorem 4.5.

**Theorem 7.5.** *Let $H(0)=0$, $r_H=1$ and $H\in MDA(\Psi_\gamma)$. (a) If* (7.11) *holds, then*

$$\hat{I}(x)=\int_{1-x}^{1}\phi(x/y)dH(y)=(1+o(1))C\frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)}x^{\beta}\overline{H}(1-x),\quad (x\downarrow 0).$$

*(b) If* (7.12) *holds, then*

$$\hat{J}(x)=\int_{1-x}^{1}y^{-1}g(x/y)dH(y)=(1+o(1))c\frac{\Gamma(\beta)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma)}x^{\beta-1}\overline{H}(1-x),\quad (x\downarrow 0).$$

Proof of Theorem 7.5 For (a) simply observe that if $1-x<y<1$, then

$$\phi\left(1-\frac{y-x}{y}\right)=(1+o(1))C(y-x)^{\beta}y^{-\beta},\quad (x\downarrow 0).$$

Hence $\hat{I}(x)$ is asymptotically equal to $C\Gamma(\beta+1)(\mathcal{J}_{\beta+1,p_{-\beta}}H)(1-x)$, and the assertion follows from Theorem 7.3. Similarly, $\hat{J}(x)$ is asymptotically equal to $c\Gamma(\beta)(\mathcal{J}_{\beta,p_{-\beta}}H)(1-x)$. □

Proof of Theorem 4.5 If $H\in MDA(\Psi_\gamma)$, then (4.22) and (4.23) follow from Theorem 7.5.

Proof of Lemma 5.2 It follows from (4.5) that

$$(y+t/w(y))^p=(1+o(1))py^{p-1}/w(y)=(1+o(1))\left(y^p+(t/w_p(y^p))\right),$$

and hence that the necessary and sufficient condition (4.4) applied to $F$ is equivalent to

$$\lim_{y\to\infty}\boldsymbol{P}\{X^p>y^p+t/w_p(y^p)|X^p>y^p\}=e^{-t}.$$

Setting $x=y^p$ shows this is equivalent to $F_p\in MDA(\Lambda,w_p)$. □

Proof of Theorem 5.1 Let $G_2$ and $H_2$ denote the distribution functions of $U^2$ and $R^2$, respectively. It follows from (5.2), Lemma 5.2 and Theorem 4.1 that

$$H\in MDA(\Lambda,w)\text{ iff }H_2\in MDA(\Lambda,w_2)\text{ iff }G_2\in MDA(\Lambda,w_2)\text{ iff }G\in MDA(\Lambda,w).$$

This, together with Theorem 12.3.3 in Berman (1992) implies that (5.3) holds if $G\in MDA(\Lambda,w)$, i.e. (a) is valid.

By the same reasoning, (b) follows if we prove it assuming $H\in MDA(\Lambda,w)$ and $H$ has a density function $h$. Observing that $Z_x$ has the same distribution as $\sqrt{1-\rho^2}S_2|S_1=x$, we set $\rho=0$ without loss of generality. In this case (following the example of Abdous et al. (2005)) we can use the equivalent representation $(U,V)=(I_1\sqrt{B},I_2\sqrt{(1-B^2)})$, where $I_1$, $I_2$ and $B$ are independent, the $I_j=\pm1$ with equal probability, and $B\sim$ beta$(1/2,1/2)$. The joint density function $f(u,v)$ of $(U,v)$ is radially symmetric and a routine computation yields

$$f(u,v)=\left[2\pi\sqrt{u^2+v^2}\right]^{-1}h\left(\sqrt{u^2+v^2}\right).$$

It is more expedient to work directly in terms of $h_2(z)=(2\sqrt{z})^{-1}h\left(\sqrt{z}\right)$, whence

$$f(u,v)=\pi^{-1}h_2\left(u^2+v^2\right).$$

Integration with respect to $v$ and using the substitution $y=v^2$ gives the marginal density function of $U$,

$$f_U(u)=\frac{1}{\pi}\int_0^\infty h_2(y+u^2)y^{-1/2}dy,$$

and hence the density function of $Z_x$ is

$$f(v|x):=f(x,v)/f_U(x)=\frac{h_2\left(x^2+v^2\right)}{\int_0^\infty h_2(y+v^2)y^{-1/2}dy},$$

valid for real $v$ and $x > 0$. Note that the distribution of $Z_x$ is symmetric about zero.

Let $t > 0$ and replace $v$ with $t/c(x)$ in this density function. Since $c^2(x) = 2w_2(x^2)$, the density function of $c(x)Z_x$ is the function of $s = x^2$ given by

$$\zeta(t|x) = \frac{h_2(s + t^2/2w_2(s))}{\sqrt{2w_2(s)} \int_0^\infty h_2(s+y)y^{-1/2}dy}. \tag{7.16}$$

Divide the numerator and denominator of the right-hand side by $w_2(s)\overline{H}_2(s)$. Since $h_2(s) = (1+o(1))w_2(s)\overline{H}_2(s)$, it follows from Lemma 5.2, and (4.4) and (4.6) applied to $H_2$, that the numerator term obtained from (7.16) converges to $\exp(-t^2/2)$, as $x \to r_h$.

Next, making the substitution $z = yw_2(s)$ in the integral at (7.16), the denominator term obtained from the division operation is

$$\frac{\sqrt{2}}{w_2(s)\overline{H}_2(s)} \int_0^\infty h_2(s + z/w_2(s))z^{-1/2}dz = \sqrt{2}\boldsymbol{E}\left\{W_s^{-1/2}\right\},$$

where $W_s$ is as defined in the proof of Theorem 7.1 ($s$ replacing $x$ there). The moment convergence theorem ensures that

$$\lim_{x \to r_H} \boldsymbol{E}\left\{W_s^{-1/2}\right\} = \boldsymbol{E}\left\{W^{-1/2}\right\} = \Gamma(1/2) = \sqrt{\pi}.$$

Combining these limits shows that $\zeta(t|x)$ converges to the standard Gaussian density function, and the assertion follows. □

# References


[1] Abdous, B., Fougères, A.-L., and Ghoudi, K. (2005) Extreme behaviour for bivariate elliptical distributions. *The Canadian Journal of Statistics* **33**, 317-334.

[2] Abdous, B., Fougères, A.-L., Ghoudi, K., and Soulier, P. (2007) Estimation of bivariate excess probabilities for elliptical models. (*www.arXiv:math.ST/0611914v2*).

[3] Berman, M.S. (1983) Sojourns and extremes of Fourier sums and series with random coefficients. *Stoch. Proc. Appl.* **15**, 213–238.

[4] Berman, M.S. (1992) *Sojourns and Extremes of Stochastic Processes*. Wadsworth & Brooks/ Cole, Boston.

[5] Beutner, E., and Kamps, U. (2008a) Models of ordered data and products of beta random variables. In: B.C. Arnold, N. Balakrishnan, J.M. Sarabia and R. Mnguez (Hrsg.), Advances in Mathematical and Statistical Modeling (S.). Boston: Birkhuser.

[6] Beutner, E., and Kamps, U. (2008b) Random contraction and random dilation of generalized order statistics. *Comm. Stat. Theory and Meth.* To appear.

[7] Breiman, L. (1965). On some limit theorems similar to the arc-sin law. *Theory of Probability and its Applications*, **10**, 323–331.

[8] Cambanis, S., Huang, S., and Simons, G. (1981) On the theory of elliptically contoured distributions. *J. Multivariate Analysis.* **11**,3, 368–385.

[9] Cline, D.B.H., and Samorodnitsky, G. (1994) Subexponentiality of the product of independent random variables. *Stoch. Proc. Appl.* **49**, 1, 75–98.

[10] de Haan, L., and Ferreira, A. (2006) *Extreme Value Theorey. An Introdution.* Springer, New York.

[11] Diebolt, J., Gardes, L., Girard, S., Guillou, A. (2007) Bias-reduced extreme quantiles estimators of Weibull distributions. To appear in *J. Stat. Plan. Inf.*

[12] Dufresne, D. (2007) beta products with complex parameters. Preprint.

[13] Falk, M., Hüsler, J., and Reiss R.-D. (2004) *Laws of Small Numbers: Extremes and Rare Events.* DMV Seminar **23**, Second Edition, Birkhäuser, Basel.

[14] Fang, K.-T., Kotz, S., and Ng, K.-W. (1990) *Symmetric Multivariate and Related Distributions.* Chapman and Hall, London, United Kingdom.

[15] Feller, W. (1971) *An Introduction to Probability Theory and its Applications, Vol. II, 2nd. ed.* New York: Wiley.

[16] Galambos, J., and Simonelli, I. (2004) *Products of random variables.* New York: Marcel Dekker Inc.

[17] Gardes, L., and Girard, S. (2006) Comparison of Weibull tail-coefficient estimators. *REVSTAT - Statistical Journal*, **4**,2,163–188.

[18] Hashorva, E. (2006) Gaussian approximation of conditional elliptical random vectors. *Stoch. Models.* **22**,3, 441-457.

[19] Hashorva, E. (2007) Extremes of $L_p-$norm asymptotically spherical distributions. *Albanian J. Math.*, **1**,3, 157–172.

[20] Hashorva, E, and Kotz, S. (2009) On the strong Kotz approximation of Dirichlet random vectors. *Statistics*, to appear.

[21] Hashorva, E, Kotz, S., and Kume, A. (2007) $L_p$-norm generalised symmetrised Dirichlet distributions. *Albanian J. Math.* **1**,1, 31–56.

[22] Heffernan, J.E., and Tawn, J.A. (2004) A conditional approach for multivariate extremes. *J. R. Statist. Soc.* B **66**, 497–546.

[23] Jessen, A.H., and Mikosch, T. (2006) Regularly varying functions. *Publications de l'Institut Mathmatique, Nouvelle Série*, **80**(94) , 171–192.

[24] Kotz, S., Balakrishnan, N., and Johnson, N.L. (2000) *Continuous Multivariate Distributions*. Second Edition, Wiley, New York.

[25] Li, D., and Peng, L. (2009) Goodness-of-fit test for tail copulas modeled by elliptical copulas. *Statist. Prob. Letters*, To appear. (Available on line.)

[26] Miller, K.S., and Ross, B. (1993) *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York.

[27] Nadarajah, S. (2005) Sums, products and ratios of generalized beta variables. *Statist. Papers* **47**, 69–90.

[28] Nadarajah, S., Kotz, S. (2005) On the product and ratio of gamma and beta random variables. *Allgemeines Statistisches Archiv*, **89**, 435–449.

[29] Pakes, A.G. (1996) Length biasing and laws equivalent to the log-normal. *J. Math. Anal. Appl.* **197**, 825–854.

[30] Pakes, A.G. (2004) Convolution equivalence and infinite divisibility. *J. Appl. Probab.* **41**, 407–424.

[31] Pakes, A.G. (2007) Structure of Stieltjes classes of moment-equivalent probability laws. *J. Math. Anal. Appl.* **326**, 2, 1268–1290.

[32] Pakes, A.G., and Navarro, J. (2007) Distributional characterizations through scaling relations. *Aust. N. Z. J. Stat.* **49**,2, 115-135.

[33] Pakes, A.G., and Steutel, F.W. (1997) On the number of records near the maximum. *The Austral. J. Statist.* **39**, 172–192.

[34] Reiss, R-D. (1989) *Approximate Distributions of Order Statistics: With Applications to Nonparametric Statistics*. Springer, New York.

[35] Resnick, S.I. (2008) *Extreme Values, Regular Variation and Point Processes.* Springer, New York.

[36] Tang, Q. (2006) The subexponentiality of products revisited. *Extremes*, **9**, (3-4), 231–241.

[37] Tang, Q. (2008) From light tails to heavy tails through multiplier. *Extremes*, **11**, 4, 379–391.

[38] Tang, Q. and Tsitsiashvili, G. (2003) Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stochastic Process. Appl.* **108**, 2, 299–325.

[39] Tang, Q. and Tsitsiashvili, G. (2004) Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. *Adv. in Appl. Probab.* **36**, 4, 1278–1299.